

Canonical variables for steep planar water waves over nonuniform beds

V. P. Ruban*

Landau Institute for Theoretical Physics, 2 Kosygin Street, 119334 Moscow, Russia

(Received 15 February 2010; published 6 May 2010)

An explicit expression in terms of canonical variables is obtained for the Hamiltonian functional determining the fully nonlinear dynamics of two-dimensional potential flows of an ideal fluid with a free surface over an arbitrary nonuniform depth. The canonically conjugate variables are derived from the previously developed noncanonical conformal description of water waves over a strongly undulating bottom [V. P. Ruban, Phys. Rev. E **70**, 066302 (2004)]. Also an alternative approach to the problem is discussed, which gives weakly nonlinear Hamiltonian models of different orders, both for free-surface waves and for interfacial waves.

DOI: [10.1103/PhysRevE.81.056303](https://doi.org/10.1103/PhysRevE.81.056303)

PACS number(s): 47.35.Bb, 47.15.km, 47.10.Df, 47.35.Lf

I. INTRODUCTION

For many phenomena related to the science of ocean waves, it is necessary to study fully nonlinear regimes in the dynamics of free water surface. For instance, rogue waves are highly nonlinear structures attracting much interest over past years [1]. Breaking waves on a beach give another example of nonlinear water wave dynamics. For some problems the simplified mathematical formulation can be used when the fluid is treated as inviscid and incompressible, and only potential flows with a constant pressure above the free surface are considered. Such class of ideal flows is known to possess the Hamiltonian structure [2], where the pair of canonically conjugate functions is formed, for example, by the vertical displacement $\eta(\mathbf{r}, t)$ of the free surface from the mean horizontal level and by the boundary distribution $\psi(\mathbf{r}, t)$ of the velocity potential, with \mathbf{r} being the position in the horizontal plane and t being the time. Unfortunately, the variables η and ψ are not suitable for highly nonlinear waves, since there is no explicit expression for the Hamiltonian functional in terms of η and ψ , valid for arbitrary wave steepness. For two-dimensional (2D) flows (in xy plane) this difficulty can be avoided, and exact and compact equations of motion are possible with the help of so-called conformal variables originally introduced by Ovsyannikov in early 1970s [3,4]. The conformal description was combined with a variational formalism in 1990s by Zakharov and co-workers, for the infinite depth [5] and for the case of flat horizontal bed [6]. Subsequent development of this method was done in Refs. [7–10]. The general idea of the approach is to use a time-dependent conformal mapping $x+iy=z(w, t)$ which “straightens” the flow domain, with an analytic function $z(w, t)$ depending on a complex variable $w=u+iv$. The free surface and the bottom correspond to some fixed values v_s and v_b of the curvilinear coordinate v (in the case of infinite depth $v_s=0$ and $v_b=-\infty$, while in the case of finite depth without loss of generality one can take $v_s=1$ and $v_b=0$), and thus the conformally invariant 2D Laplace equation for the velocity potential is easily solved. Basic unknown dynamical quantities, such as the boundary value ψ of the velocity potential and the vertical coordinate Y along the free

surface, in this description are functions of u and t . Explicit expressions for time derivatives $\psi_t(u, t)$ and $Y_t(u, t)$ can be derived from variational Euler-Lagrange equations with the help of a linear integral relation between $Y(u, t)$ and $X(u, t)$, taking place due to the analyticity [5,6].

Nontrivial bottom topography is known to affect wave motion. In order to include into the fully nonlinear model that influence, the conformal description was generalized by the present author to the case of static nonuniform bed [11], for time-dependent bottom boundary [12], and for planar flows with constant vorticity over nonuniform bed [13]. Even for weakly nonlinear regimes, conformal coordinates in the problems with bottom topography are more preferable than the Cartesian coordinates, though in that case a somewhat different approach is employed (see, for example, Refs. [11,14,15], and Sec. III below). A step toward using conformal variables for three-dimensional (3D) highly nonlinear potential flows with a free surface was done in the paper [16], where for long-crested waves an asymptotic expansion of the Hamiltonian functional was suggested in the powers of a small parameter corresponding to the squared ratio of a typical wavelength λ_0 to a long typical distance l_q along wave crests stretching in the transversal horizontal direction q .

Equations of motion in the conformal description contain integral operators which are diagonal in the Fourier representation. Therefore in the numerical simulations a fast Fourier transform (FFT) can be used (see, for example, the website [17] about a modern and efficient FFT library). Important numerical results concerning planar rogue waves on infinitely deep water have been obtained by this method in Refs. [18–21]. Nonlinear waves over inhomogeneous static and time-dependent beds were simulated in the conformal variables in Refs. [11,12]. The most recent application of this method can be found in Refs. [22,23], where the phenomenon of so-called water-wave gap solitons over spatially periodic beds was studied. The weakly 3D conformal theory [16] has been recently successfully applied to simulate long-crested freak waves at infinitely deep water [24–27].

It should be noted that in most cited papers noncanonical conformal variables were used as they are quite sufficient for numerical simulations. The exceptions are the analytical works [8,9], where a pair of canonical conformal variables has been employed, however in the case of infinite depth only. For uneven beds, the canonical pair was not derived

*ruban@itp.ac.ru

until now. In the present work, this gap in the theory will be filled and canonical conformal variables together with an explicit Hamiltonian will be suggested for 2D ideal potential flows over an arbitrary nonuniform bottom profile (in Sec. II). But we will see the obtained exact Hamiltonian contains a strongly nonlocal linear operator which at long scales is similar to the integrating operator $\hat{\sigma}_u^{-1}$. The presence of such operator makes the exact canonical model somewhat difficult for analysis. Therefore an alternative approach how to introduce canonical variables for waves over strongly undulating bottom will be also considered (in Sec. III), which is free of the above-mentioned technical difficulty. Unfortunately, the alternative approach does not allow us to obtain an exact Hamiltonian, but approximations of different orders for weakly nonlinear waves are possible. In this paper, the calculations will be made up to the fifth order. In addition, it will be shown that the same approach is applicable also in the theory of internal waves in the Boussinesq limit within the two-layer model (see, e.g., Refs. [28–33] and references therein about two-layer models).

II. EXACT HAMILTONIAN FOR WAVES OVER NONUNIFORM DEPTH

It is appropriate to mention here that if the given bottom boundary is nonflat, then the conformal mapping $z(w, t)$ can be represented as the composition $z(w, t) = Z[\zeta(w, t), t]$, where $Z(\zeta, t)$ is a known analytic function [$Z(\zeta)$ does not depend on time if the bed is static, which case is assumed in this work]. The intermediate unknown analytic function $\zeta(w, t)$ takes purely real values $\gamma(u, t)$ at the real axis: $\zeta(u + 0i, t) = \gamma(u, t)$ and $\gamma_u(u, t) > 0$. As a result, the bed profile is given in the parametric form $X^{[b]} + iY^{[b]} = Z(\gamma)$, where $-\infty < \gamma < +\infty$, while the shape of free surface is determined parametrically by the formula

$$X^{[s]} + iY^{[s]} = Z[\xi(u, t)], \quad \xi(u, t) = \zeta(u + i, t). \quad (1)$$

Since $\zeta(u + i, t)$ is the analytic continuation of a real function $\gamma(u, t)$ from the real axis, a linear integral relation takes place between the real and the imaginary parts of function $\xi(u, t)$,

$$\xi(u, t) = e^{-\hat{k}}\gamma(u, t) = (1 + i\hat{R})\rho(u, t), \quad (2)$$

where $\hat{k} = -i\hat{\partial}_u$ is the differential operator, $e^{-\hat{k}}$ is the operator making the analytic continuation, $\rho(u, t) = [\cosh \hat{k}]\gamma(u, t)$ is another unknown real function, and $\hat{R} = i[\tanh \hat{k}]$ is a linear antisymmetric operator which is diagonal in the Fourier representation,

$$\hat{R}\rho(u, t) = \int i[\tanh k]\rho_k(t)e^{iku} \frac{dk}{2\pi}, \quad (3)$$

with $\rho_k(t) = \int \rho(u, t)e^{-iku} du$ being the Fourier transform of the function $\rho(u, t)$. The inverse operator is $\hat{T} = \hat{R}^{-1} = -i[\coth \hat{k}]$. For long waves, when $|k| \ll 1$, operator \hat{R} is similar to $\hat{\partial}_u$, while \hat{T} is similar to integrating operator $\hat{\sigma}_u^{-1}$. Explicit formulas for the kernels of operators \hat{R} and \hat{T} in u representation are given, for instance, in Ref. [11].

A simple way to obtain the required canonical pair is the following. We first recall the expression for the Lagrangian functional describing water waves over a static nonuniform bed in terms of the conformal variables (see Ref. [11]),

$$\mathcal{L} = \int \psi|Z'(\xi)|^2(\rho_u \hat{R}\rho_t - \rho_t \hat{R}\rho_u) du - \mathcal{H}. \quad (4)$$

Here $Z'(\xi) \equiv dZ/d\xi$, and the Hamiltonian \mathcal{H} is the total energy of the system—the kinetic energy plus the potential energy in the vertical gravitational field g (surface-tension effects can be also included, but we do not consider them in this paper),

$$\mathcal{H} = \frac{1}{2} \int \psi \hat{K} \psi du + \frac{g}{2} \int [\text{Im} Z(\xi)]^2 \text{Re}[Z'(\xi)\xi_u] du, \quad (5)$$

where $\hat{K} \equiv \hat{k} \tanh \hat{k}$ is a Hermitian operator.

Let us now choose function $\rho(u, t)$ as the generalized canonical coordinate. Using the antisymmetric property of operator \hat{R} , we rewrite Eq. (4) as follows:

$$\mathcal{L} = \int \{-\hat{R}[\psi|Z'(\xi)|^2\rho_u] - \psi|Z'(\xi)|^2\hat{R}\rho_u\}\rho_t du - \mathcal{H}. \quad (6)$$

It is clear that the Lagrangian will take the canonical form $\mathcal{L} = \int \mu \rho_t du - \mathcal{H}$ if we define the canonical momentum μ by the relation written below

$$\mu = -\hat{R}[\psi|Z'(\xi)|^2\rho_u] - \psi|Z'(\xi)|^2\hat{R}\rho_u. \quad (7)$$

Using this equation, it is now necessary to express ψ through the canonical variables μ and ρ and then substitute the result into Eq. (5). Generally speaking, this is not a trivial task in view of the presence of integral operator \hat{R} . Fortunately, the coefficients of the equation are very special, and therefore the solution can be found in exact form. To solve the integral Eq. (7) with respect to ψ , we rewrite it in the form

$$\text{Re}\{[1 + i\hat{R}](-i\psi|Z'(\xi)|^2\xi_u + \mu)\} = 0. \quad (8)$$

The above equation allows us to conclude that

$$-i\psi|Z'(\xi)|^2\xi_u + \mu = -i(1 - i\hat{R})f, \quad (9)$$

where f is some real function. Since $\bar{\xi}_u = (1 - i\hat{R})\rho_u$ (the overline means the complex conjugate quantity), we can multiply Eq. (9) by $i\bar{\xi}_u$ and use a general formula

$$[(1 - i\hat{R})f_1][\overline{(1 - i\hat{R})f_2}] = (1 - i\hat{R})f_3, \quad (10)$$

where f_1 , f_2 , and f_3 are real functions, and $f_3 = f_1 f_2 - (\hat{R}f_1)(\hat{R}f_2)$. As the result, we have

$$\psi|Z'(\xi)\xi_u|^2 + i\mu\bar{\xi}_u = (1 - i\hat{R})\tilde{f}, \quad (11)$$

where \tilde{f} is another real function. Taking the imaginary part of Eq. (11), we obtain $\mu\rho_u = -\hat{R}\tilde{f}$, and thus $\tilde{f} = -\hat{T}(\mu\rho_u)$. Then the real part gives us the required formula

$$\psi = \frac{-\hat{T}(\mu\rho_u) - \mu\hat{R}\rho_u}{|\hat{\partial}_u Z([1 + i\hat{R}]\rho)|^2}. \quad (12)$$

Now we are able to write explicit and exact expression for the Hamiltonian functional in terms of the canonical variables ρ and μ ,

$$\begin{aligned} \mathcal{H} = & \frac{1}{2} \int \left[\frac{\hat{T}(\mu\rho_u) + \mu\hat{R}\rho_u}{|\hat{\partial}_u Z([1 + i\hat{R}]\rho)|^2} \right] \hat{K} \left[\frac{\hat{T}(\mu\rho_u) + \mu\hat{R}\rho_u}{|\hat{\partial}_u Z([1 + i\hat{R}]\rho)|^2} \right] du \\ & + \frac{g}{2} \int [\text{Im } Z([1 + i\hat{R}]\rho)]^2 \text{Re}[\hat{\partial}_u Z([1 + i\hat{R}]\rho)] du, \quad (13) \end{aligned}$$

which is the central result of this paper. The obtained formula is rather cumbersome, and it is unclear yet if some canonical transformation will be able to reduce this Hamiltonian to a simpler form. Further serious work on the problem how to simplify the Hamiltonian (13) is needed.

The corresponding canonical equations of motion are $\rho_t = \delta\mathcal{H}/\delta\mu$ and $-\mu_t = \delta\mathcal{H}/\delta\rho$, where the variational derivatives should be calculated from Eq. (13) according to the well-established general rules. Let us introduce a short-hand notation

$$N \equiv |\hat{\partial}_u Z(\xi)|^{-2} \hat{K} \left[\frac{\hat{T}(\mu\rho_u) + \mu\hat{R}\rho_u}{|\hat{\partial}_u Z(\xi)|^2} \right]. \quad (14)$$

Then the variational derivatives are

$$\frac{\delta\mathcal{H}}{\delta\mu} = (\hat{R}\rho_u)N - \rho_u\hat{T}N, \quad (15)$$

$$\begin{aligned} \frac{\delta\mathcal{H}}{\delta\rho} = & \hat{\partial}_u[\hat{R}(\mu N) + \mu\hat{T}N] - 2 \text{Re}[(1 - i\hat{R})\{\hat{T}(\mu\rho_u) \\ & + \mu\hat{R}\rho_u\}NZ''(\xi)/Z'(\xi)] + 2 \text{Re}[(1 - i\hat{R})\hat{\partial}_u\{\hat{T}(\mu\rho_u) \\ & + \mu\hat{R}\rho_u\}N/\xi_u] + g \text{Im}[(1 - i\hat{R})\{\text{Im}(Z(\xi))|Z'(\xi)|^2\bar{\xi}_u\}]. \quad (16) \end{aligned}$$

III. ALTERNATIVE APPROACH

An essential difficulty of the derived fully nonlinear canonical model is the presence of strongly nonlocal operator \hat{T} . Therefore it makes sense to consider alternative approaches to the problem of canonical variables for waves over uneven bed. Below we shall focus on the approximate method suggested in Ref. [11], where curvilinear conformal coordinates u and v are static. In this approach, the conformal mapping “straightens” the bottom boundary but not the free surface, in contrast to the exact description used in the previous section. In Ref. [11], the corresponding Hamiltonian functional was derived up to the third order in canonical variables. Here we develop this method more systematically and calculate the fourth- and fifth-order approximations.

We now introduce (static) curvilinear coordinates $u(x, y)$ and $v(x, y)$, where real function $v(x, y)$ obeys the Laplace

equation $v_{xx} + v_{yy} = 0$ with the boundary conditions $v = -1$ at the given arbitrary nonuniform bottom and $v = 0$ at $y = 0$. Function $u(x, y)$ is taken harmonically conjugate for $v(x, y)$. The inverse conformal mapping $x + iy = z(u + iv)$ satisfies the condition $\text{Im } z(u + 0i) = 0$, so a real function $x(u) = z(u + 0i)$ parametrizes the unperturbed free surface $y = 0$. The bed profile is given in the parametric form $X^{[b]} + iY^{[b]} = z(u - i)$, while the shape of the free boundary is determined through an unknown real function $V(u, t)$ by the complex equality

$$X^{[s]} + iY^{[s]} = z[u + iV(u, t)]. \quad (17)$$

The free-boundary value of the velocity potential is $\psi(u, t) = \varphi[u, V(u, t), t]$, where the potential $\varphi(u, v, t)$ obeys 2D Laplace equation $\varphi_{uu} + \varphi_{vv} = 0$ in the flow domain $-1 \leq v \leq V(u, t)$, with the bottom boundary condition $\varphi_v(u, -1, t) = 0$. Consequently, a general form of the Lagrangian functional in this description is the following:

$$\mathcal{L} = \int \psi |z'(u + iV)|^2 V_t du - \mathcal{H}\{V, \psi\}. \quad (18)$$

The Hamiltonian functional $\mathcal{H}\{V, \psi\} = \mathcal{K}\{V, \psi\} + \mathcal{P}\{V\}$, where $\mathcal{K}\{V, \psi\}$ is the kinetic energy and $\mathcal{P}\{V\}$ is the potential energy. Functional $\mathcal{P}\{V\}$ is relatively simple,

$$\begin{aligned} \mathcal{P} = & \frac{g}{2} \int [\text{Im } z(u + iV)]^2 \text{Re}[\hat{\partial}_u z(u + iV)] du \\ = & \frac{g}{2} \int \left\{ V^2 x'^3 + V^4 \left[\frac{(x'^2 x'')'}{4} - \frac{5}{6} x'^2 x''' \right] \right\} du + \mathcal{O}(V^6), \quad (19) \end{aligned}$$

with $x' = dx(u)/du$, $x'' = d^2x(u)/du^2$ and so on. As to the kinetic energy $\mathcal{K}\{V, \psi\}$, it cannot be represented in an exact form, but its expansion $\mathcal{K} = \mathcal{K}^{[2]} + \mathcal{K}^{[3]} + \mathcal{K}^{[4]} + \dots$ in powers of (supposedly small) functions V and ψ is possible. We note that the equations determining the 2D velocity potential $\varphi(u, v, t)$ are formally identical to the equations for the velocity potential $\varphi(x, y, t)$ in Cartesian coordinates over a straight horizontal bottom at $y = -1$. The kinetic energy is determined by the same integral in both cases,

$$\mathcal{K}\{V, \psi\} = \frac{1}{2} \int du \int_{-1}^{V(u)} (\varphi_u^2 + \varphi_v^2) dv. \quad (20)$$

In view of this equivalence, the expansion of $\mathcal{K}\{V, \psi\}$ is calculated in the same standard manner as for weakly nonlinear waves over flat bottom in Cartesian coordinates (compare to expansion of $\mathcal{K}\{\eta, \psi\}$ in Appendix B of Ref. [23]),

$$\mathcal{K}^{[2]} = \frac{1}{2} \int \psi \hat{K} \psi du, \quad (21)$$

$$\mathcal{K}^{[3]} = \frac{1}{2} \int V[(\psi_u)^2 - (\hat{K}\psi)^2] du, \quad (22)$$

$$\mathcal{K}^{[4]} = \frac{1}{2} \int [\psi \hat{K} V \hat{K} V \hat{K} \psi + V^2 (\hat{K}\psi) \psi_{uu}] du, \quad (23)$$

$$\begin{aligned} \mathcal{K}^{[5]} = & \frac{1}{2} \int \left[\frac{V^3}{6} (\hat{K}\psi) \hat{K} \psi_{uu} - \psi \hat{K} V \hat{K} V \hat{K} V \hat{K} \psi - \frac{V^3}{3} (\psi_{uu})^2 \right. \\ & \left. - V^2 (\hat{K} V \hat{K} \psi) \psi_{uu} - \frac{V^2}{2} (\hat{K}\psi) \partial_u^2 (V \hat{K}\psi) \right] du. \end{aligned} \quad (24)$$

As a consequence, the bed nonuniformity does not affect $\mathcal{K}\{V, \psi\}$ in any order. This is an essential technical advantage of the conformal coordinates in this problem.

The form of Lagrangian (18) allows us to treat $\psi(u)$ as the canonical momentum if the corresponding canonical coordinate $\chi(u)$ is defined in the following way:

$$\chi(u) = \int_0^V |z'(u + iv)|^2 dv. \quad (25)$$

Since we consider small values V , the above equation gives us an expansion of χ in odd powers of V ,

$$\chi = x'^2 V + [(x''')^2 - x' x'''] V^3 / 3 + \mathcal{O}(V^5). \quad (26)$$

From here we can express the noncanonical variable V through the canonical variable χ :

$$V = \frac{\chi}{x'^2} \left[1 + \left(\frac{x' x''' - (x'')^2}{x'^6} \right) \frac{\chi^2}{3} + \mathcal{O}(\chi^4) \right]. \quad (27)$$

Now we substitute the above expression into $\mathcal{H}\{V, \psi\}$ and collect there terms of the same order. After simplification, we obtain the Hamiltonian functional $\mathcal{H}\{\chi, \psi\} = \mathcal{H}^{[2]} + \mathcal{H}^{[3]} + \dots$ up to the fifth order,

$$\mathcal{H}^{[2]} = \frac{1}{2} \int \left[\psi \hat{K} \psi + \frac{g}{x'} \chi^2 \right] du, \quad (28)$$

$$\mathcal{H}^{[3]} = \frac{1}{2} \int [(\psi_u)^2 - (\hat{K}\psi)^2] \frac{\chi}{x'^2} du, \quad (29)$$

$$\begin{aligned} \mathcal{H}^{[4]} = & \frac{1}{2} \int \left[\psi \hat{K} \frac{\chi}{x'^2} \hat{K} \frac{\chi}{x'^2} \hat{K} \psi + \frac{\chi^2}{x'^4} (\hat{K}\psi) \psi_{uu} \right] du \\ & + \frac{g}{12} \int \left[\frac{1}{2} \frac{x'''}{x'^6} - \frac{(x'')^2}{x'^7} \right] \chi^4 du, \end{aligned} \quad (30)$$

$$\begin{aligned} \mathcal{H}^{[5]} = & \frac{1}{6} \int [(\psi_{uu})^2 - (\hat{K}\psi)^2] \left[\frac{x' x''' - (x'')^2}{x'^8} \right] \chi^3 du \\ & + \frac{1}{12} \int \left[\frac{\chi^3}{x'^6} (\hat{K}\psi) \hat{K} \psi_{uu} - 6 \psi \hat{K} \frac{\chi}{x'^2} \hat{K} \frac{\chi}{x'^2} \hat{K} \frac{\chi}{x'^2} \hat{K} \psi \right. \\ & - 2 \frac{\chi^3}{x'^6} (\psi_{uu})^2 - 6 \frac{\chi^2}{x'^4} \left(\hat{K} \frac{\chi}{x'^2} \hat{K} \psi \right) \psi_{uu} \\ & \left. - 3 \frac{\chi^2}{x'^4} (\hat{K}\psi) \partial_u^2 \left(\frac{\chi}{x'^2} \hat{K} \psi \right) \right] du. \end{aligned} \quad (31)$$

We do not write here the corresponding canonical equations of motion—though it is a simple exercise to calculate the variational derivatives $\delta\mathcal{H}/\delta\chi$ and $\delta\mathcal{H}/\delta\psi$, but the formulas are quite long. This weakly nonlinear model can be applied to a wide variety of problems, for instance, to study the dy-

namics of solitons propagating along a channel of variable depth or the scattering of a wave train by a strong bed non-uniformity. The obvious advantages of the model are that it is fully dispersive and there is no principal limitations on the bottom profiles. However, it is not clear at the moment what is the largest value of wave amplitude which is still well described by this approximate Hamiltonian. The answer perhaps depends on many details. This question requires further investigations.

Weakly dispersive (shallow-water) regime can take place if the depth is slowly varying, $|x''|/x' \ll 1$. In that regime one can reduce the Hamiltonian to a purely local form by using the long-wave expansion for the integral operator $\hat{K} = -\partial_u^2 - \partial_u^4/3 + \dots$ and then apply some analytical methods. Otherwise, the fully dispersive equations of motion can be easily simulated on computer with the help of FFT routines since all linear operators are diagonal in Fourier representation.

It is interesting to note that the weakly nonlinear approach developed here has also another important application, namely, for the theory of internal waves in the ocean. Internal waves in many cases are described by the simplified two-layer model [28–33], assuming the system to consist of two layers of immiscible irrotational fluids with different densities ($\rho_1 = 1$ in the upper layer, the equilibrium depth of this layer is h , and $\rho_2 = 1 + \epsilon$ in the lower layer). In the practically important case $\epsilon \ll 1$ (the Boussinesq limit), the free surface above the upper layer can be effectively treated as a rigid lid, and then the dynamics of interfacial waves is determined by the Lagrangian functional of the form Eq. (18), where function $V(u, t) < 0$ is a shape of the interface in the conformal coordinates. However, function ψ now has a different physical meaning, namely, $\psi = \rho_2 \psi_2 - \rho_1 \psi_1 \approx (\psi_2 - \psi_1)$, where $\psi_1(u, t)$ and $\psi_2(u, t)$ are boundary values of the two velocity potentials at the interface. As to the Hamiltonian functional for internal waves, in the Boussinesq approximation it can be calculated using the fact that $\Omega(u, v, t) = \delta[v - V(u, t)] \psi_u(u, t)$ is the singular vorticity field concentrated at the interface between the layers, and the kinetic energy is

$$\begin{aligned} \mathcal{K}_{iw} = & -\frac{1}{2} \iint G(u_1, v_1, u_2, v_2) \\ & \times \Omega(u_1, v_1) \Omega(u_2, v_2) du_1 dv_1 du_2 dv_2, \end{aligned} \quad (32)$$

where $G(u_1, v_1, u_2, v_2)$ is the Green's function of the Laplace operator in the stripe of unit width, with zero boundary conditions. As the result, the Hamiltonian is expressed by the following formula:

$$\begin{aligned} \mathcal{H}_{iw} = & \frac{\epsilon g}{2} \int [\text{Im } z(u + iV) + h]^2 \text{Re}[\partial_u z(u + iV)] du \\ & + \frac{1}{4\pi} \iint \ln \left| \frac{\sinh \left[\frac{\pi}{2} (u_2 - u_1 + iV_2 + iV_1) \right]}{\sinh \left[\frac{\pi}{2} (u_2 - u_1 + iV_2 - iV_1) \right]} \right| \\ & \times \psi'(u_1) \psi'(u_2) du_1 du_2, \end{aligned} \quad (33)$$

where $V_1 = V(u_1)$, $V_2 = V(u_2)$, and $\psi' = \partial_u \psi$. If the upper layer

is shallow compared with the characteristic depth [that is $|V(u, t)| \ll 1$], then approximate formula (27) can be used, and its substitution into Eq. (33) will give us an approximate Hamiltonian in terms of canonical variables. Otherwise, one has to invert somehow the full Eq. (25).

To conclude, in this paper two different methods have been presented how to solve the theoretical problem of canonical Hamiltonian description of nonlinear planar water waves over nonuniform depth. The first method is fully nonlinear, but the corresponding Hamiltonian functional contains a strongly nonlocal operator. The second way is a further development of the weakly nonlinear approach suggested in

Ref. [11], and it provides a very clear and compact procedure how to derive weakly nonlinear canonical equations of motion containing only “well-behaved” nonlocal operators.

ACKNOWLEDGMENTS

These investigations were supported by RFBR (Grants No. 09-01-00631 and No. 07-01-92165), by the “Leading Scientific Schools of Russia” Grant No. 6885.2010.2, and by the Program “Fundamental Problems of Nonlinear Dynamics” from the RAS Presidium.

-
- [1] C. Kharif and E. Pelinovsky, *Eur. J. Mech. B/Fluids* **22**, 603 (2003).
- [2] V. E. Zakharov, *J. Appl. Mech. Tech. Phys.* **2**, 190 (1968).
- [3] L. V. Ovsyannikov, *Sib. Branch USSR Acad.* **15**, 104 (1973).
- [4] L. V. Ovsjannikov, *Arch. Mech.* **26**, 407 (1974).
- [5] A. I. Dyachenko, E. A. Kuznetsov, M. D. Spector, and V. E. Zakharov, *Phys. Lett. A* **221**, 73 (1996).
- [6] A. I. Dyachenko, V. E. Zakharov, and E. A. Kuznetsov, *Plasma Phys. Rep.* **22**, 829 (1996).
- [7] A. I. Dyachenko, *Dokl. Math.* **63**, 115 (2001).
- [8] A. I. Dyachenko, Y. V. Lvov, and V. E. Zakharov, *Physica D* **87**, 233 (1995).
- [9] Y. V. Lvov, *Phys. Lett. A* **230**, 38 (1997).
- [10] W. Choi and R. Camassa, *J. Eng. Mech.* **125**, 756 (1999).
- [11] V. P. Ruban, *Phys. Rev. E* **70**, 066302 (2004).
- [12] V. P. Ruban, *Phys. Lett. A* **340**, 194 (2005).
- [13] V. P. Ruban, *Phys. Rev. E* **77**, 037302 (2008).
- [14] A. Nachbin, *SIAM J. Appl. Math.* **63**, 905 (2003).
- [15] W. Artiles and A. Nachbin, *Phys. Rev. Lett.* **93**, 234501 (2004).
- [16] V. P. Ruban, *Phys. Rev. E* **71**, 055303(R) (2005).
- [17] <http://www.fftw.org/>
- [18] V. E. Zakharov, A. I. Dyachenko, and O. A. Vasilyev, *Eur. J. Mech. B/Fluids* **21**, 283 (2002).
- [19] A. I. Dyachenko and V. E. Zakharov, *JETP Lett.* **81**, 255 (2005).
- [20] V. E. Zakharov, A. I. Dyachenko, and A. O. Prokofiev, *Eur. J. Mech. B/Fluids* **25**, 677 (2006).
- [21] A. I. Dyachenko and V. E. Zakharov, *JETP Lett.* **88**, 307 (2008).
- [22] V. P. Ruban, *Phys. Rev. E* **77**, 055307(R) (2008).
- [23] V. P. Ruban, *Phys. Rev. E* **78**, 066308 (2008).
- [24] V. P. Ruban and J. Dreher, *Phys. Rev. E* **72**, 066303 (2005).
- [25] V. P. Ruban, *Phys. Rev. E* **74**, 036305 (2006).
- [26] V. P. Ruban, *Phys. Rev. Lett.* **99**, 044502 (2007).
- [27] V. P. Ruban, *Phys. Rev. E* **79**, 065304(R) (2009).
- [28] V. D. Djordjevic and L. G. Redekopp, *J. Phys. Oceanogr.* **8**, 1016 (1978).
- [29] W. Choi and R. Camassa, *J. Fluid Mech.* **396**, 1 (1999).
- [30] L. A. Ostrovsky and J. Grue, *Phys. Fluids* **15**, 2934 (2003).
- [31] W. Craig, P. Guyenne, and H. Kalisch, *Commun. Pure Appl. Math.* **58**, 1587 (2005).
- [32] V. Vlasenko and N. Stashchuk, *J. Phys. Oceanogr.* **36**, 1959 (2006).
- [33] A. R. de Zarate and A. Nachbin, *Commun. Math. Sci.* **6**, 385 (2008).